INTRODUCTION TO PROBABILITY MODELS

Lecture 22

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CONTINUOUS RANDOM VARIABLE

- **Definition:**a random variable that can take on any value in a range.
- For example:
  - Blood pressure
  - The height of a 8 year old
  - Mile per gallon of a car
  - GPA

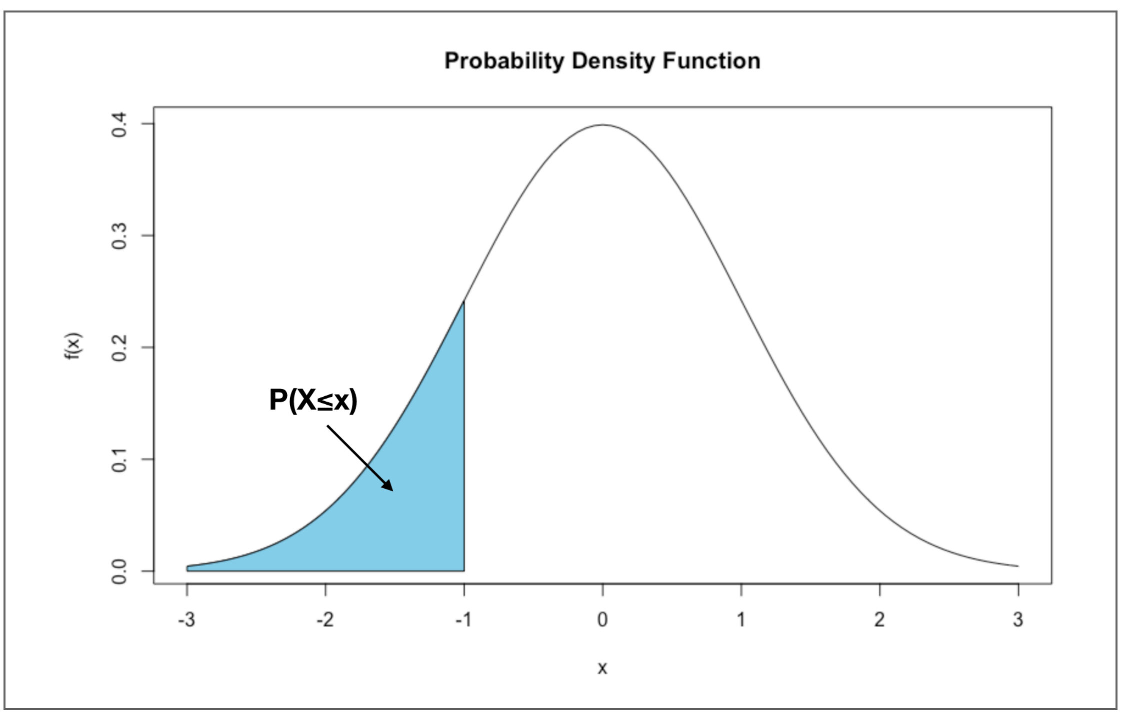
PROBABILITY MASS FUNCTION REVISIT

- **Definition:**a function that gives the probability that a **discrete** random variable is exactly equal to some value.
- For every x, $0 \leq p_X(x) \leq 1$
- $\sum_x p_X(x) = 1$
- For example, $X \sim Binomial(n = 4, p = 0.3)$, $P(X \leq 2) = P_X(0) + P_X(1) + P_X(2)$

PROBABILITY DENSITY FUNCTION

In order to find probabilities of continuous random variables, we can no longer use a PMF, because the probabilities are no longer at points, they are over regions. Instead we have a Probability Density Function, or PDF, $f_X(x)$, which looks more like a traditional function over a region. If you are given a PDF, the probability can be calculated as:

$$P(X \leq a) = \int_{-\infty}^{a} f_X(x)dx$$



PROPERTIES OF A CONTINOUS PDF

If $f_X(x)$ is a probability density function(PDF), then

- $f_X(x) \geq 0$ for all values of X in its support
- $\int_{-\infty}^{\infty} f_X(x)dx = 1$
- $P(X \leq a) = \int_{\infty}^{a} f_X(x)dx$
- $P(a \leq X \leq b) = \int_{a}^{b} f_X(x)dx$

EXAMPLE 1

Let $f_X(x) = 0.25x$ for $1 \leq x \leq 3$ and 0 otherwise

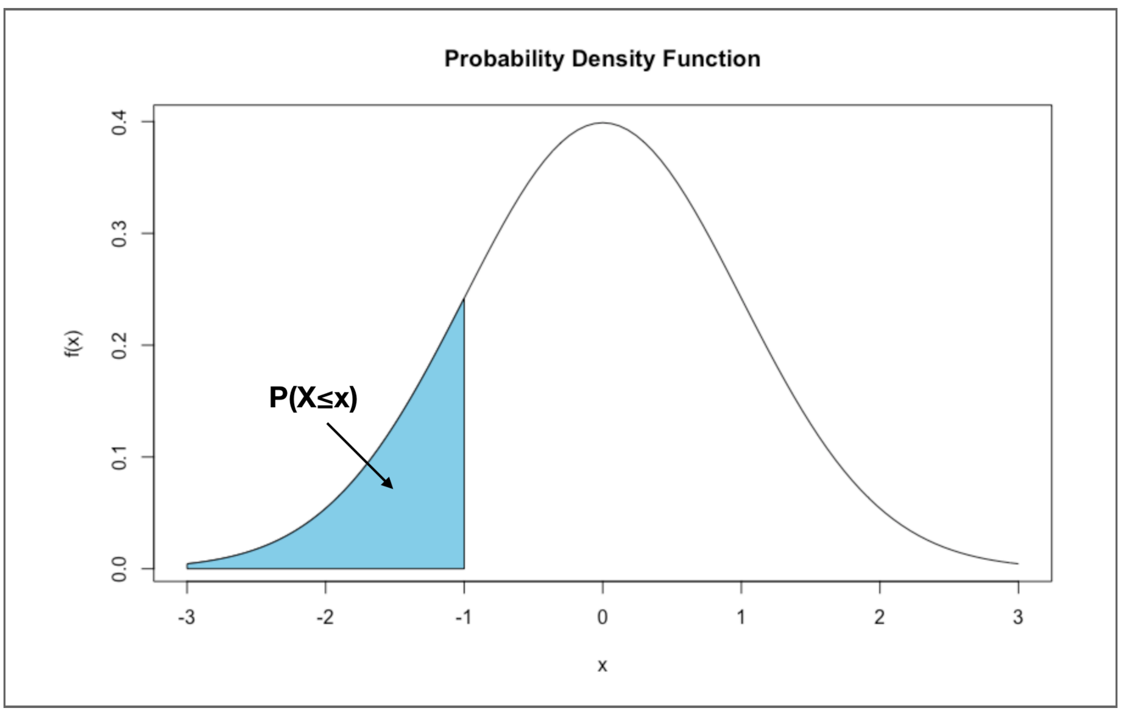
1. Make of graph of the PDF
2. Is X more likely to be in the interval $[1, 2]$ or $[2, 3]$

CUMULATIVE DENSITY FUNCTION

The cumulative distribution function F(x) for a continuous random variable X is defined for every number x by

$$F_X(x) = P(X \leq x) = \int_{-\infty}^{x} f_X(t)dt$$

For each x, $F_(x)$ is the area under the density curve to the left of x.



EXPECTED VALUE AND VARIANCE FOR A CONTINUOUS RANDOM VARIABLE

X is a continuous random variable with PDF $f_X(x)$

- **Expected Value:**
  $$E[X] = \int_{-\infty}^{+\infty} x f_X(x) dx$$
- **Variance:**
  $$\begin{aligned} Var(X) &= E[(E - E[X])^2] \\ &= E[X^2] - E[X]^2 \\ &= \int_{-\infty}^{+\infty} x^2 f_X(x) dx - (\int_{-\infty}^{+\infty} x f_X(x) dx)^2 \end{aligned}$$

PERCENTILES

X is a continuous random variable with PDF $f_X(x)$ and CDF $F_X(x)$

- **Definition:** p_{th} percentile of X means a value k at which $P(X < k) = p\%$ or $F_X(k) = p\%$
- For example, 10_{th} percentile of X means a value k at which $P(X < k) = 0.10$ or $F_X(k) = 0.10$
- Quantile:
  - 1_{st} quantile: 25_{th} percentile
  - 2_{nd} quantile, Median: 50_{th} percentile
  - 3_{rd} quantile: 75_{th} percentile

EXAMPLE 2

Let X represent the diameter in inches of a circular disk cut by a machine. Let $f_X(x) = c(4x - x^2)$ for $1 \leq x \leq 4$ and 0 otherwise.

1. Find the value of c that makes this a valid PDF.
2. Find the expected value and variance of X
3. Find the CDF $F_X(x)$
4. Find the probability that X is within 0.5 inches of the average diameter.
5. What is the third quartile of X?

EXAMPLE 3

Suppose that a continuous random variable, X, has the probability density function (PDF) given below:

$$f_X(x) = \begin{cases} \frac{3}{2}x, 0 \leq x \leq 1 \\ \frac{1}{4}, 5 \leq x \leq 6 \\ 0, otherwise \end{cases}$$

1. What is the probability that X is equal to 4?
2. What is the probability that X is more than 4?
3. Find $F_X(5.6)$
4. Knowing that X is more than 0.8, what is the probability that X is less than 5.6?
5. What is the 85_{th} percentile of X?